Quantisation and Curved Inflation

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Cosmology!

The Cosmological Principle:

- **Homogeneity:** same at every point
- **Isotropy:** same in every direction

on large enough scales

Questions to answer:

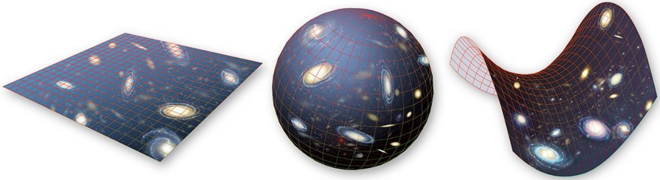
- Why is this a good assumption?
- How can we relate theory to observation?

The Basics

Given the cosmological principle, must have a spacetime of the form

$$\boxed{ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 + a^2(t)\left(\frac{1}{1-Kr^2}dr^2 + r^2 d\Omega^2\right)} \tag{1}$$

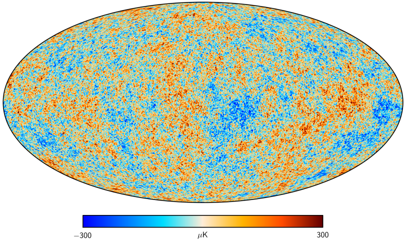
- a is called the **scale factor**
- $H = \dot{a}/a$ is called the **Hubble parameter** and measures the expansion rate of the universe.
- K is a **curvature** constant, typically rescaled to $K = -1, 0, +1$.



$K = 0, +1, -1$ **respectively**

Observation: The CMB

- Directly after the big bang, universe was super hot, super dense, opaque plasma.
- $\approx$380,000 years after BB $\longrightarrow$ universe is cool enough for hydrogen to form.
- Photons scatter off from this process unobstructed
- These photons (albiet much much colder) can be observed today!



The Cosmic Microwave Background

Average temperature: 2.725K

Problems

The Horizon Problem

- The CMB is *very* uniform
- Inhomogeneities will only grow with time.

Question: How would the very early universe be *so* uniform?

The Flatness Problem

- Curvature density diverges from it's primordial value.
- Obviously this is not observed.

Question: Why would the very early universe be incredibly flat?

Inflation!

The Idea: Make the Hubble Radius $1/aH$ **decrease**, and make this last a while.

The Method: Assume the universe is filled with a special field ϕ called **the inflaton**.

$$S = S_G + S_\phi = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \frac{1}{2}\nabla^\alpha\nabla_\alpha\phi - V(\phi)\right) \tag{2}$$

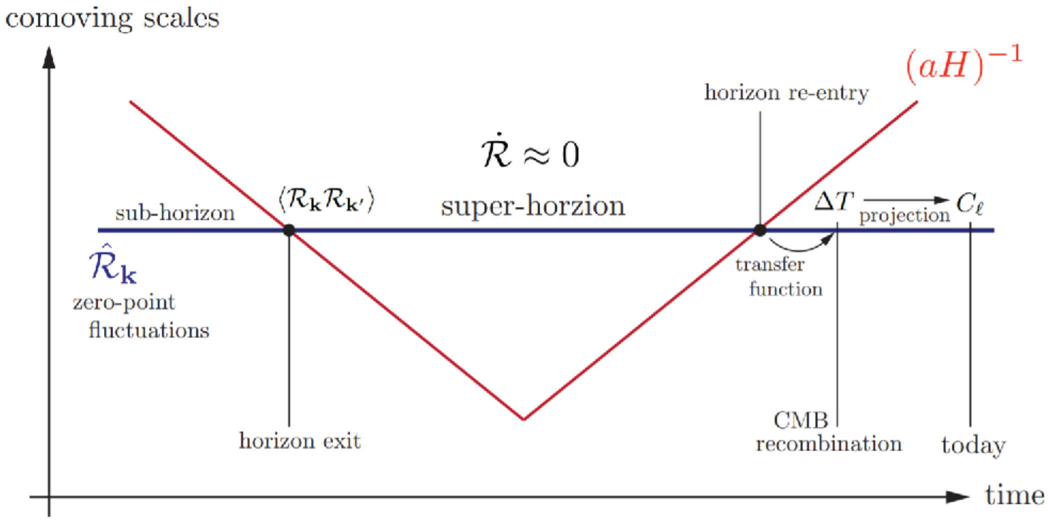
- Solves the **Horizon Problem:** energy density of inflaton dominates over inhomogeneities, and allows parts of the sky that are distant now to be in causal contact during inflation.
- Solves the **Flatness Problem:** energy density of inflaton dominates over curvature density, allowing the primordial universe to have general curvature.

The Practicalities

- Globally, the universe is homogeneous and isotropic. Everything can be modeled only by functions of time.
- Locally, we must model *spatially dependent* **perturbations** to our fields and metric.

- $\mathcal{R}$ is the **comoving curvature perturbation**
- We can use $\mathcal{R}$ to connect fluctuations predicted by inflation to anisotropies in the CMB.

The Big Picture



Why R Fails

To understand the evolution of $\mathcal{R}$, we must expand the action in terms of $\mathcal{R}$.

When $K = 0$ we have

$$S_{\mathcal{R}}^{(2)} = \frac{1}{2}\int d^4x a^3 \frac{\dot{\phi}^2}{H^2}\left(\dot{\mathcal{R}}^2 - \frac{1}{a^2}(\partial_i \mathcal{R})^2\right) \tag{3}$$

However, for a general curved universe ($K \neq 0$)

$$S_{\mathcal{R}}^{(2)} = \frac{1}{2}\int d^4x \sqrt{|c|} a^3 \frac{\dot{\phi}^2}{H^2}\left\{\frac{1}{a^2}\mathcal{R}\mathcal{D}^2\mathcal{R} + \left(\dot{\mathcal{R}} - \frac{K}{a^2}\frac{\mathcal{R}}{H}\right)\frac{\mathcal{D}^2}{\mathcal{D}^2 - K\mathcal{E}}\left(\dot{\mathcal{R}} - \frac{K}{a^2}\frac{\mathcal{R}}{H}\right)\right\}. \tag{4}$$

The Solution

Define alternative curvature perturbation variable

$$\boxed{\zeta = g\mathcal{R} - 2\frac{K}{a^2\dot{\phi}^2}\left(a^2\frac{\dot{g}H}{K} - g\right)\Phi} \tag{5}$$

This leads to a *local* action

$$S_{\zeta}^{(2)} = \frac{1}{2}\int d\eta d^3x\sqrt{c}\left((z\zeta)'' - (\nabla z\zeta)^2 + \left(3K + \frac{z''}{z}\right)(z\zeta)^2\right) \tag{6}$$

and a familiar **oscilator equation of motion** for $v = z\zeta$

$$\boxed{v'' + \left(\mathcal{D}^2 - \frac{z''}{z}\right)v = 0} \tag{7}$$

Halfway There!

Still need to set **initial conditions** for our perturbation variable.

This requires setting a **vacuum**

- Particle-less state?
- Minimum-energy / ground state?

Example: Quantum Harmonic Oscillator

- $\hat{x} = \mathbf{a} u(t) + \mathbf{a}^\dagger u^*(t)$
- Vacuum is given by lowest energy state and particle-less state: $\mathbf{a}|0\rangle = 0$

RST: A Better Way

We need a **covariant way** to set the vacuum in a *curved, expanding spacetime.*

In the *flat* case, this is much easier.

RST for Massless Scalar Field: A massless field in our spacetime has mode equation of motion

$$u'' + \left(\nabla^2 - \frac{a''}{a}\right) u = 0 \tag{8}$$

Can set the vacuum by **minimising the renormalised stress energy tensor**.

Application to Inflation: Flat case is lucky! This turns out to be *exactly* the equation of motion for $\mathcal{R}$ modes **in flat space** during inflation.

Generalising to Curved Space

Curved case is less lucky :(

It turns out we can *rescale* our curvature perturbation, and *redefine* time so that RST can be applied to ζ.

Yay! We finally have a **complete IVP**

$$0 = \ddot{\zeta} + (2\frac{\dot{z}}{z} + H)\dot{\zeta} - \frac{\mathcal{D}^2}{a^2}\zeta \tag{9}$$

$$\zeta(t_0) = \frac{1}{\sqrt{2c_s}(-\mathcal{D}^2)^{1/4}z(t_0)} \tag{10}$$

$$\dot{\zeta}(t_0) = -\frac{i\sqrt{-\mathcal{D}^2}}{a(t_0)} + H(t_0) - \frac{\dot{z}}{z}(t_0) \tag{11}$$

Mission Accomplished

We've successfully generalised the traditional inflationary calculations to **a curved spacetime**!

Next steps:

- ζ is defined by a simple but pesky differential equation for g
- Compute power spectrum for ζ and map onto Planck CMB data.