Discrete Symmetry in Lorentzian Spaces

aka Spacetime Kaleidoscopes

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Outline

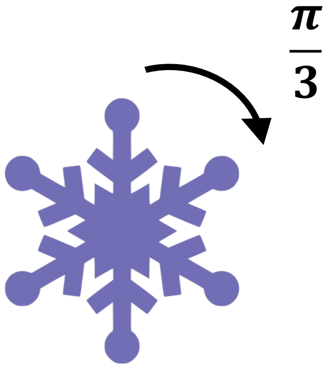
- Motivations
- **Known** Examples + Results
- **Minkowski** Examples
- Our Generalised **Definition** + **Results**

Motivation • Discrete Symmetry

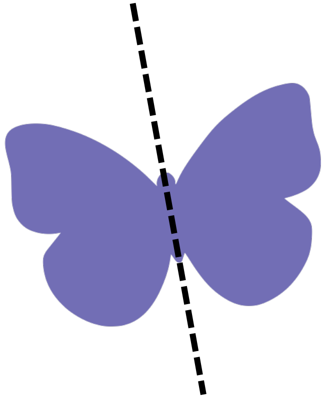
Identity

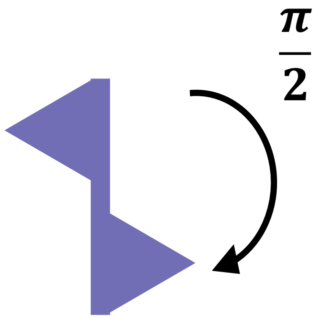
1

Rotation

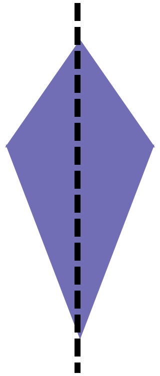


Reflection

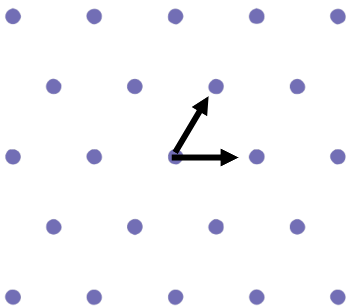




Rotation w/out Reflection



Reflection w/out Rotation

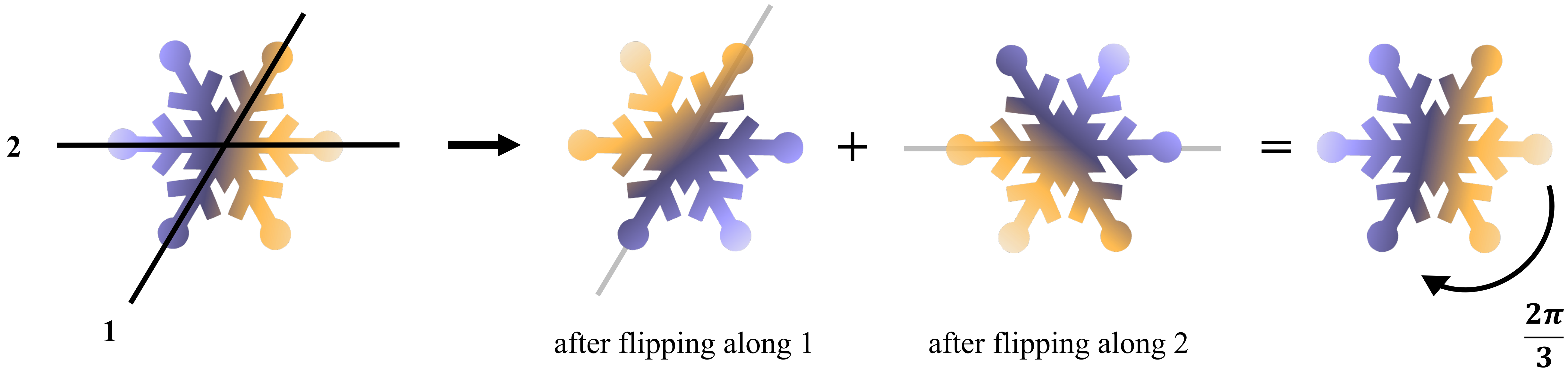


Translation

Motivation • Reflections

All above operations are generated by **reflections**.

e.g.



Motivation • Reflections

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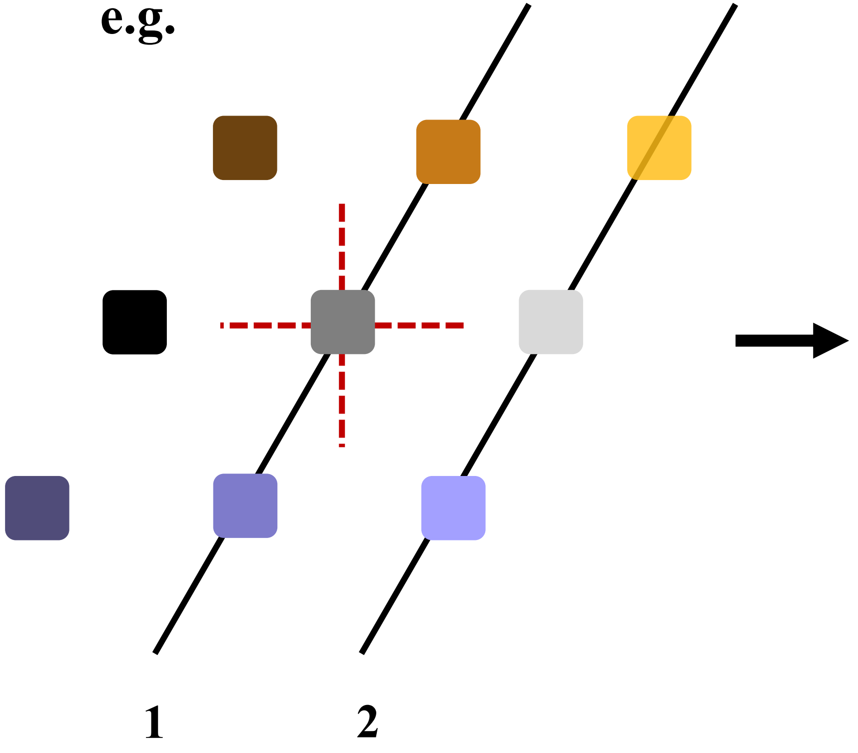
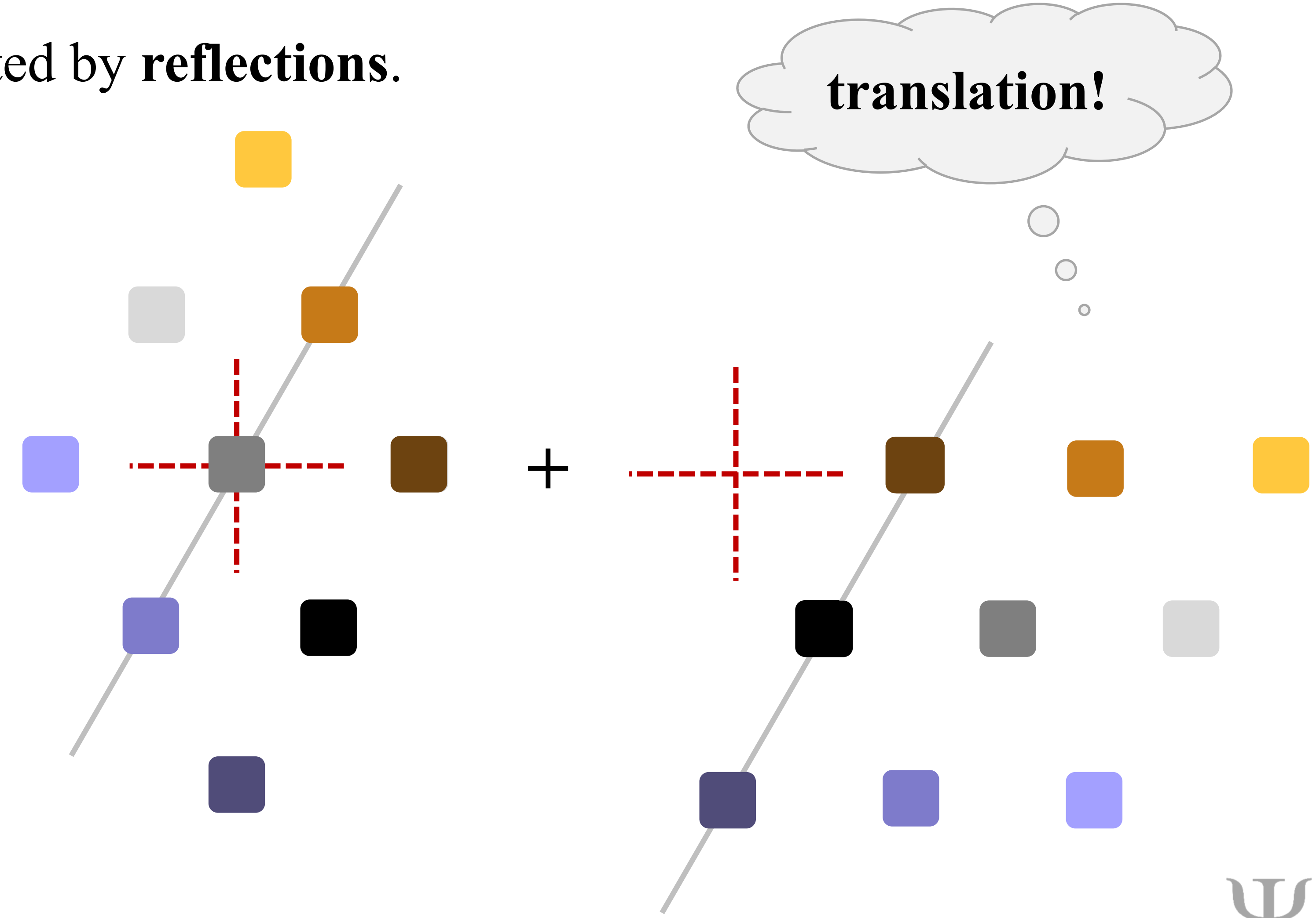
e.g.





flip along 1

flip along 2

Motivation • Reflections

Theorem (*Cartan-Dieudonné*) For a vector space V of dimension n with a nondegenerate symmetric bilinear form, any orthogonal transformation is a composition of at most n hyperplane reflections.

Motivation • Geometry ⟷ Algebra

- ***Root Vector*** = $\vec{r}$ defines hyperplane normal
- $\vec{x}$ is any Euclidean vector

good ole
Euclidean inner
product **for now**

Hyperplane | **Reflection**

$$\langle \vec{x}, \vec{r} \rangle = 0$$

$$\vec{x} \mapsto R(\vec{r})\vec{x}$$

defined by
reflection operation

$$R(\vec{r})^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - 2\frac{r_{\beta} r^{\alpha}}{\langle \vec{r}, \vec{r} \rangle}$$

$$\langle \vec{x}, \vec{r} \rangle = d$$

$$\vec{x} \mapsto R(\vec{r})\vec{x} + \frac{2d}{\langle \vec{r}, \vec{r} \rangle}\vec{r}$$

also called an
affine reflection

co-dimension
1 surfaces

Motivation • Reflection Groups

more aptly …
Coxeter Groups

Donald Coxeter
& the spherical tiling (5, 3, 2)

Classification of
Irreducible Euclidean Coxeter Groups

Lattices

Polytopes
higher dimensional analogs of polygons and polyhedra

Semi-simple finite Lie algebras

Euclidean • Crystallographic Symmetry

Lattice $\Lambda = \{\vec{x} \in V \mid \vec{x} = n_i \vec{a_i} \text{ for } n_i \in \mathbb{Z}\}$ → integer linear combinations of n linear independent basis vectors in n dimensional Euclidean space.

Symmetry operation g s.t. $g\Lambda = \Lambda$

A group Γ generated by reflections is ***crystallographic*** if it stabilizes some lattice Λ, i.e. $\gamma\Lambda = \Lambda$ for all $\gamma \in \Gamma$.

Euclidean • Lattice Symmetries

Full symmetry group

Translational symmetries
(no fixed points)

Describe using **parallel reflections**.

Point Symmetries through origin
(symmetries w/ fixed points)

Describe using **reflections through origin**.

Euclidean • Lattice Symmetries

What are reflection symmetries of lattice $\Lambda = \{\vec{x} \in V \mid \vec{x} = n_i \vec{a}_i \text{ for } n_i \in \mathbb{Z}\}$ through the origin?

- Root vector must be a **lattice vector**.
- Take root vector to be primitive – **minimal length**.

Euclidean • Lattice Symmetries

Theorem For a lattice Λ , if $\vec{v}$ is a primitive root vector of Λ then its norm $\langle\vec{v},\vec{v}\rangle$ divides $2|\Lambda|$

Euclidean • Lattice Symmetries

Theorem For a lattice Λ , if $\vec{v}$ is a primitive root vector of then its norm $\langle\vec{v},\vec{v}\rangle$ divides $2|\Lambda|$

So now we can generate all point symmetries (about origin) of lattice!

If we also include **parallel** planes with same roots passing through **non-origin points** …

… also generate translations!

= full lattice symmetry group

Euclidean • Geometry ⟷ Algebra

Hyperplane

$$\langle \vec{x}, \vec{r} \rangle = 0$$

$$\langle \vec{x}, \vec{r} \rangle = d$$

Kaleidoscope!
Collection of mirrors

Reflection

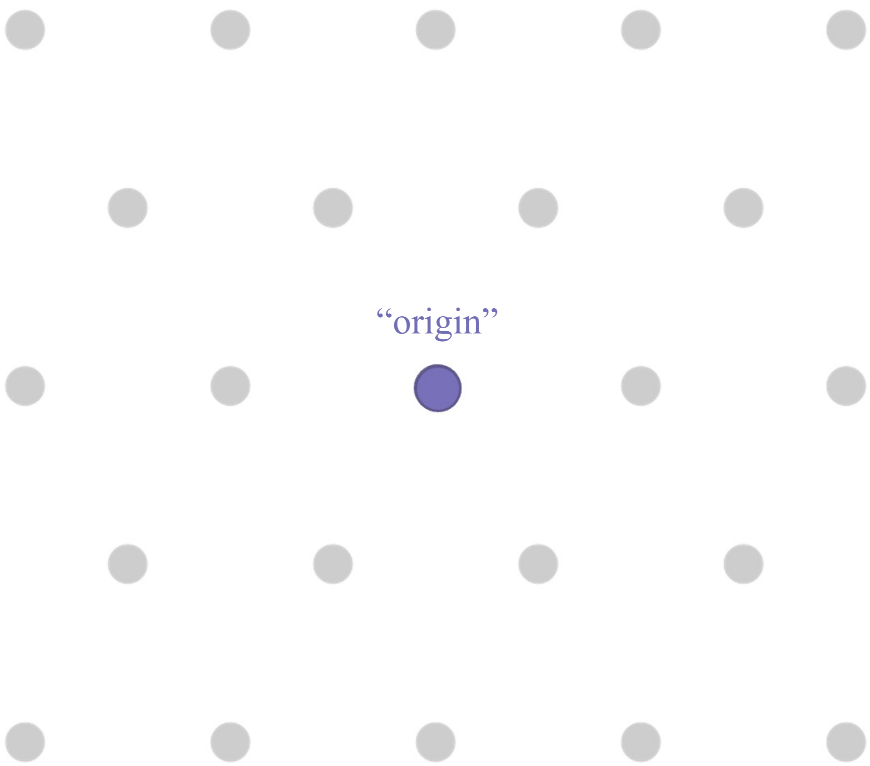
$$\vec{x} \mapsto R(\vec{r})\vec{x}$$

$$\vec{x} \mapsto R(\vec{r})\vec{x} + \frac{2d}{\langle \vec{r}, \vec{r} \rangle}\vec{r}$$

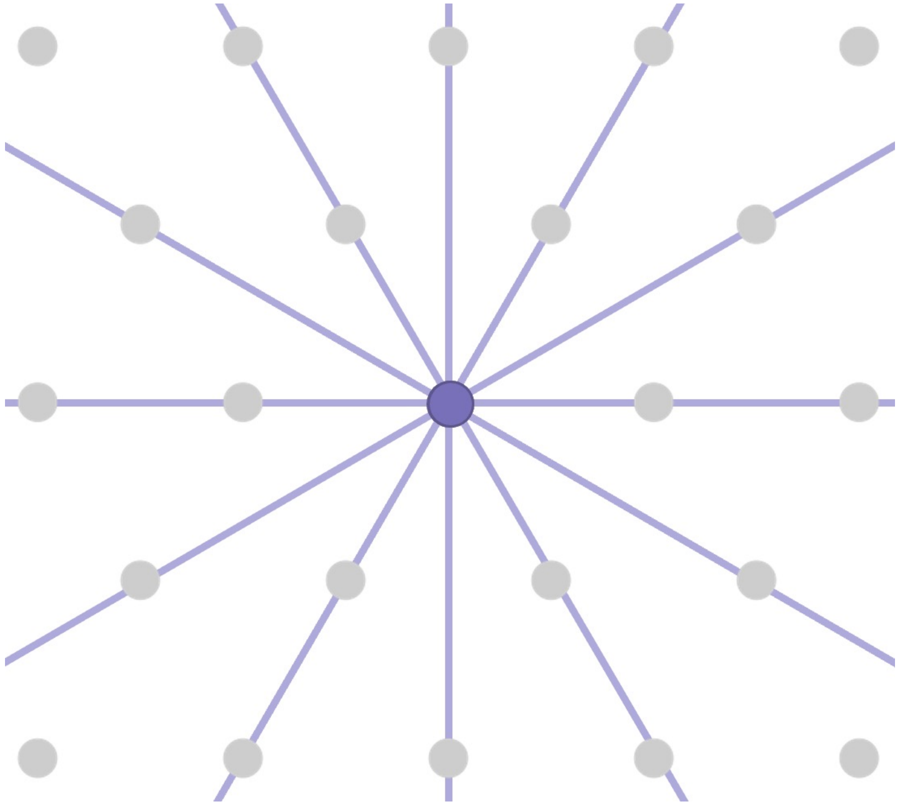
All reflection symmetries of a lattice.

Euclidean • Hexagonal Lattice

determinant = 3
so look for norm
1, 2, 3, 6 primitive roots



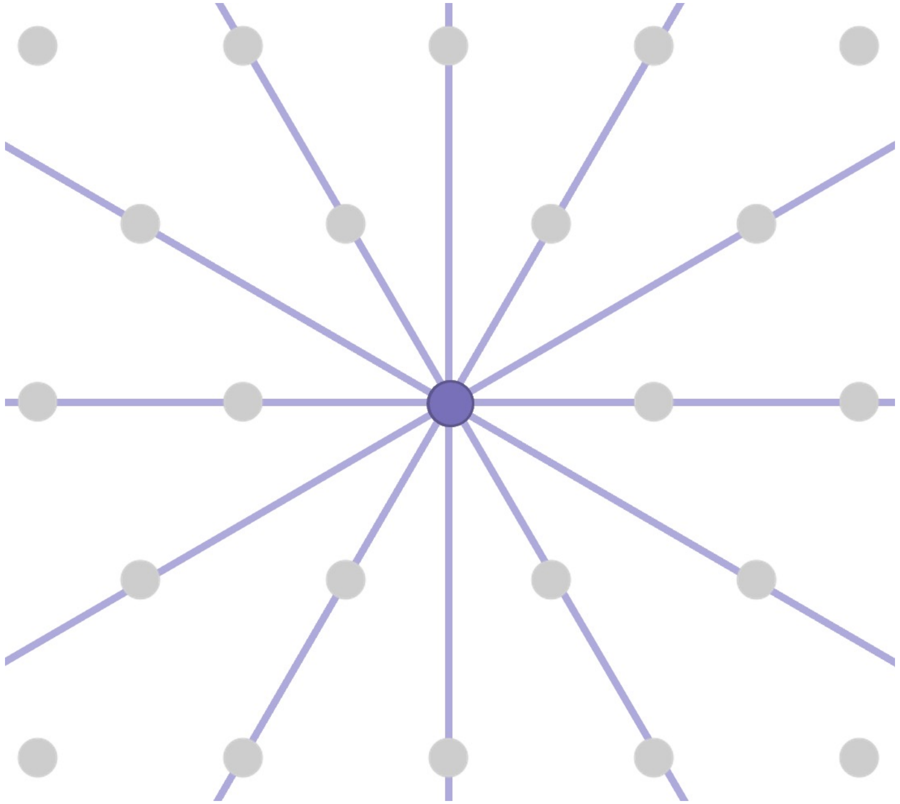
Lattice



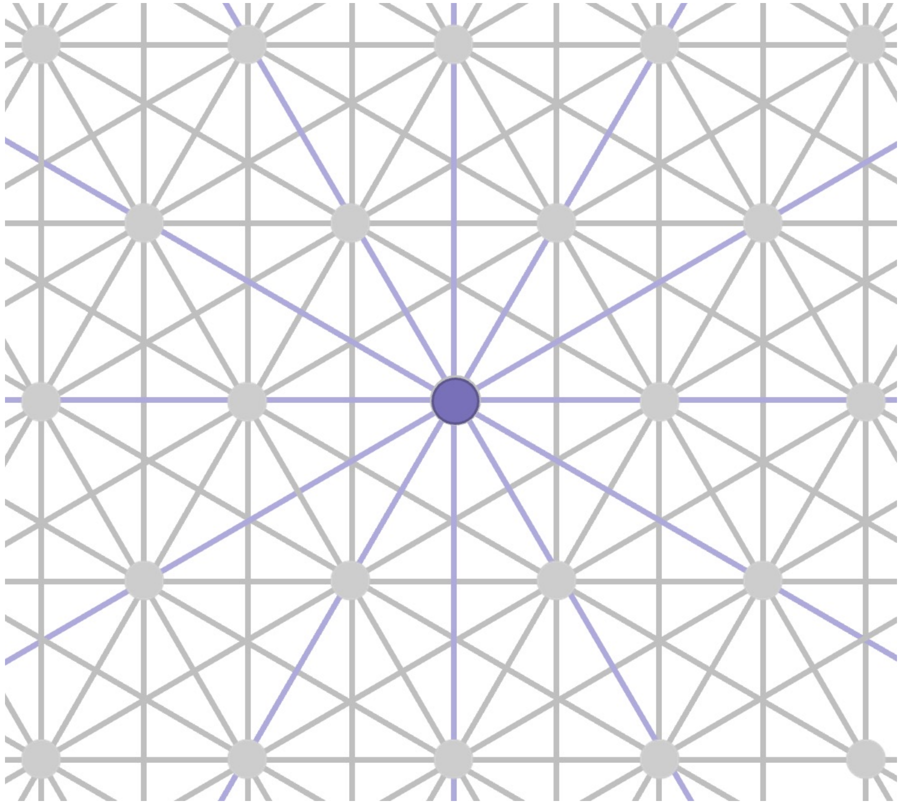
Point Reflections / Point Group

Euclidean • Hexagonal Lattice

add in affine mirrors



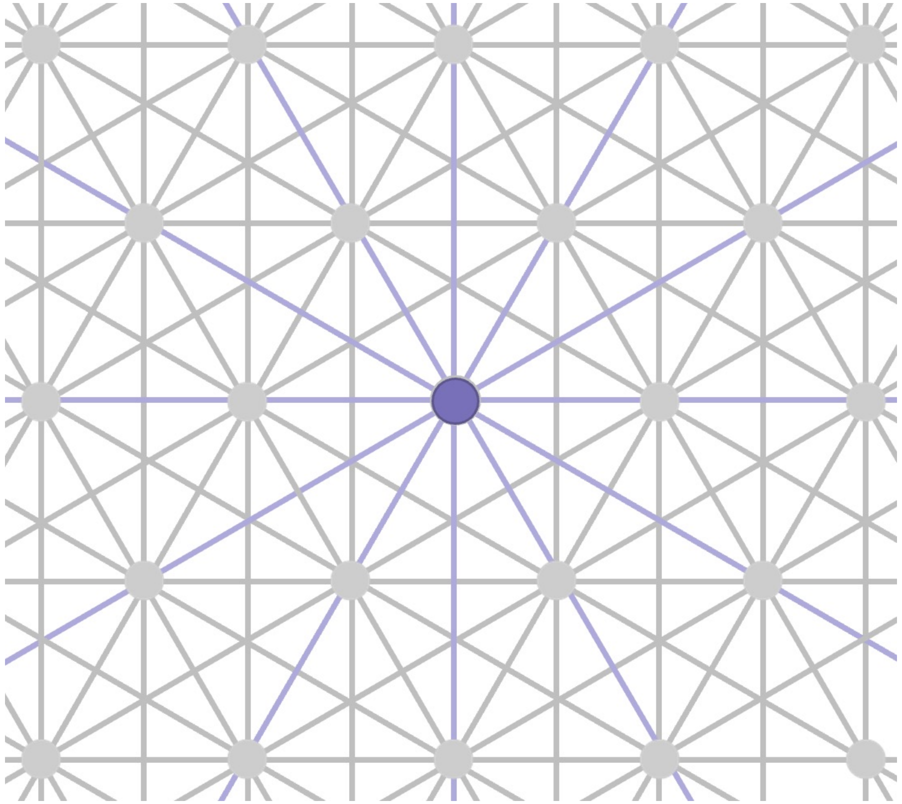
Point Group



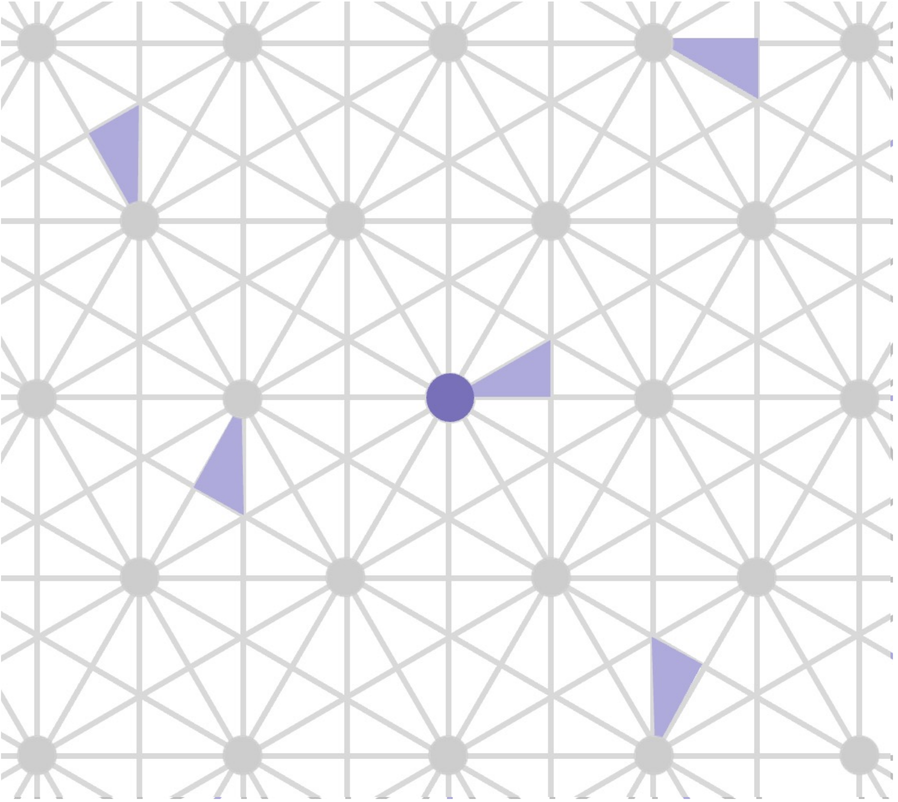
Kaleidoscope / Kaleidoscope Group

Euclidean • Hexagonal Lattice

full space now tiled by triangles
(equivalent upon kaleidoscope
reflection)



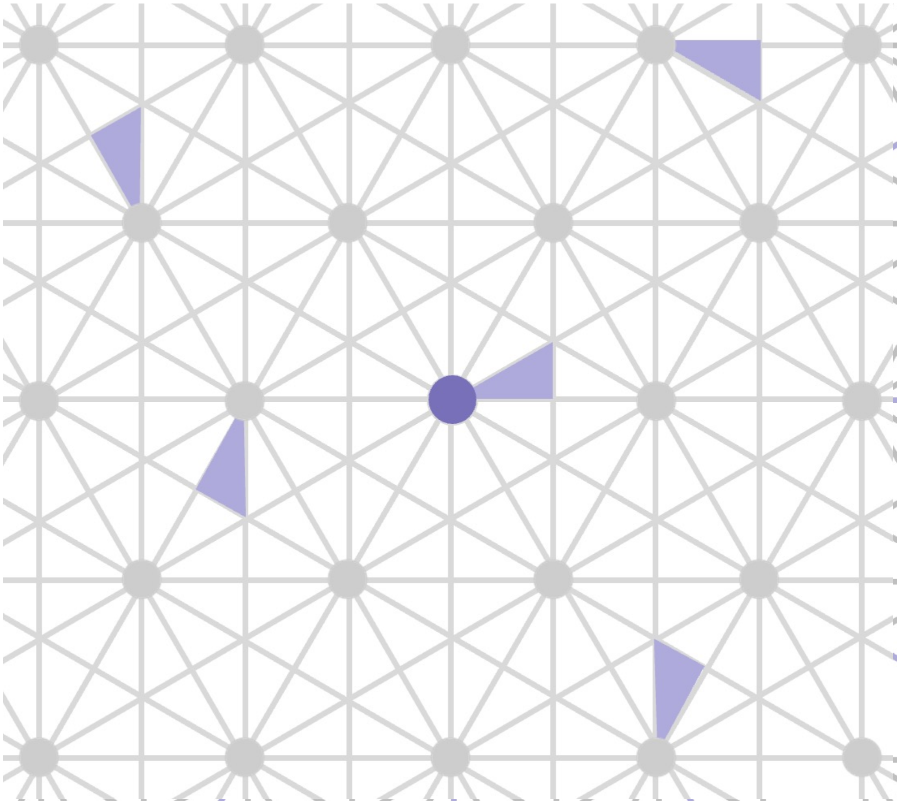
Kaleidoscope

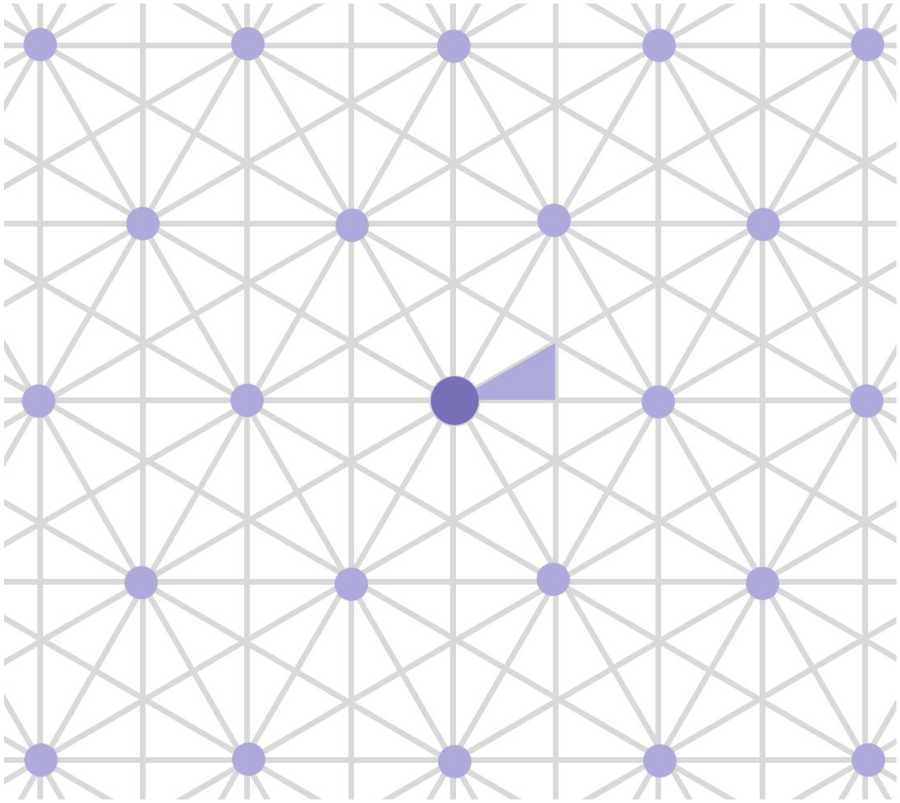


Fundamental Domain(s)

Euclidean • Hexagonal Lattice

image of point at the $\frac{\pi}{6}$ angle of fundamental domain reflected by all kaleidoscope mirrors





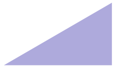
Fundamental Domain(s)

Original Lattice Regenerated!

Euclidean • Fundamental Domain

The ***fundamental domain*** defined by a group Γ acting on a space V is the orbifold (aka orbit-space manifold) defined by V/Γ

Above, Γ is a kaleidoscope group and V is Euclidean space.

→ V/Γ is the highlighted 30-60-90 triangle i.e. 

Euclidean • Fundamental Domain

The fundamental domain defined by a group Γ acting on a space V is the orbifold (aka orbit-space manifold) defined by V/Γ

Above, Γ is a kaleidoscope group and V is Euclidean space.

V/Γ is the highlighted 30-60-90 triangle i.e.

Ok … why is this important?

Fundamental Domains

Understanding fundamental domains	$\longleftrightarrow$	Classification of **irreducible finite Coxeter groups**

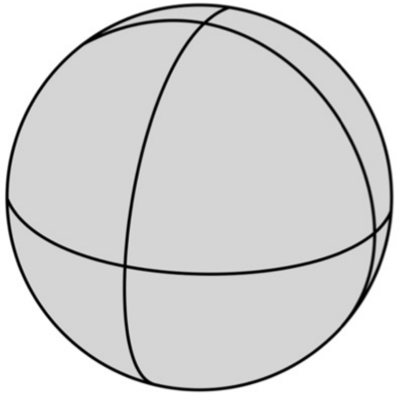
Non-Euclidean • Brief Note on Spherical Geometry

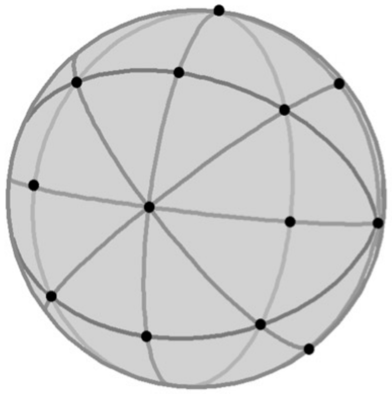
Can embed $n-1$ sphere in n dimension Euclidean space

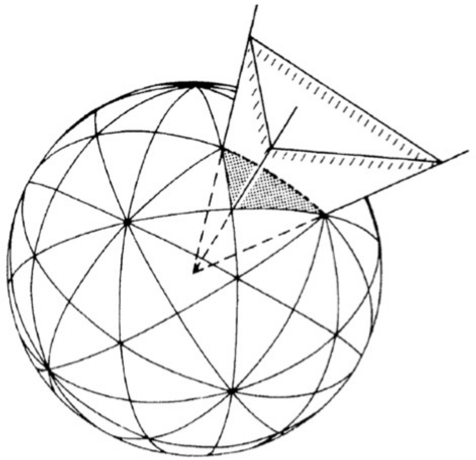
$$\mathbb{S}^{n-1} = \{\vec{v} \in \mathbb{E}^n \mid \langle \vec{v}, \vec{v} \rangle = 1\}$$

Hyperplane Mirrors / Reflection in $\mathbb{E}^n$ induce hyperplane mirrors / reflections in $\mathbb{S}^{n-1}$

spherical triangles!







Kaleidoscopic Symmetry!

Important concepts underlying above discussion.

A group Γ is ***crystallographic*** if it stabilizes some lattice Λ, i.e. $\gamma\Lambda = \Lambda$ for all $\gamma \in \Gamma$

A group Γ is ***kaleidoscopic in some space V***
if it has finite and non-zero volume orbifold V/Γ i.e. fundamental domain.

Crystallographic Symmetry!

In Euclidean space …

Crystallographic Symmetry	⟷	***Kaleidoscopic Symmetry***
is this group compatible with a lattice?		**does this group define a good tiling / discretisation?**

doesn’t make sense in spherical space!

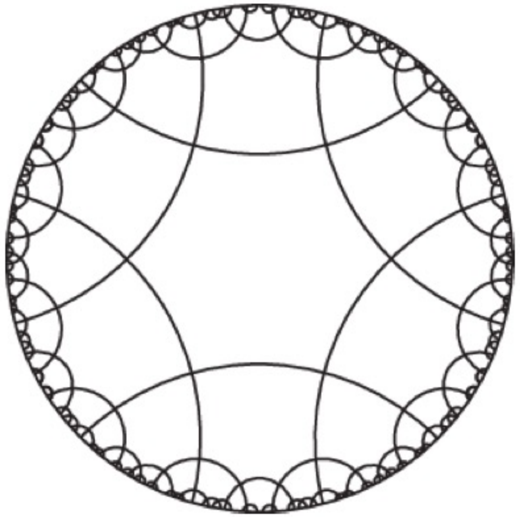
still makes sense in spherical space!

Motivation • Hyperbolic Space

Why study Lorentzian reflection symmetries?

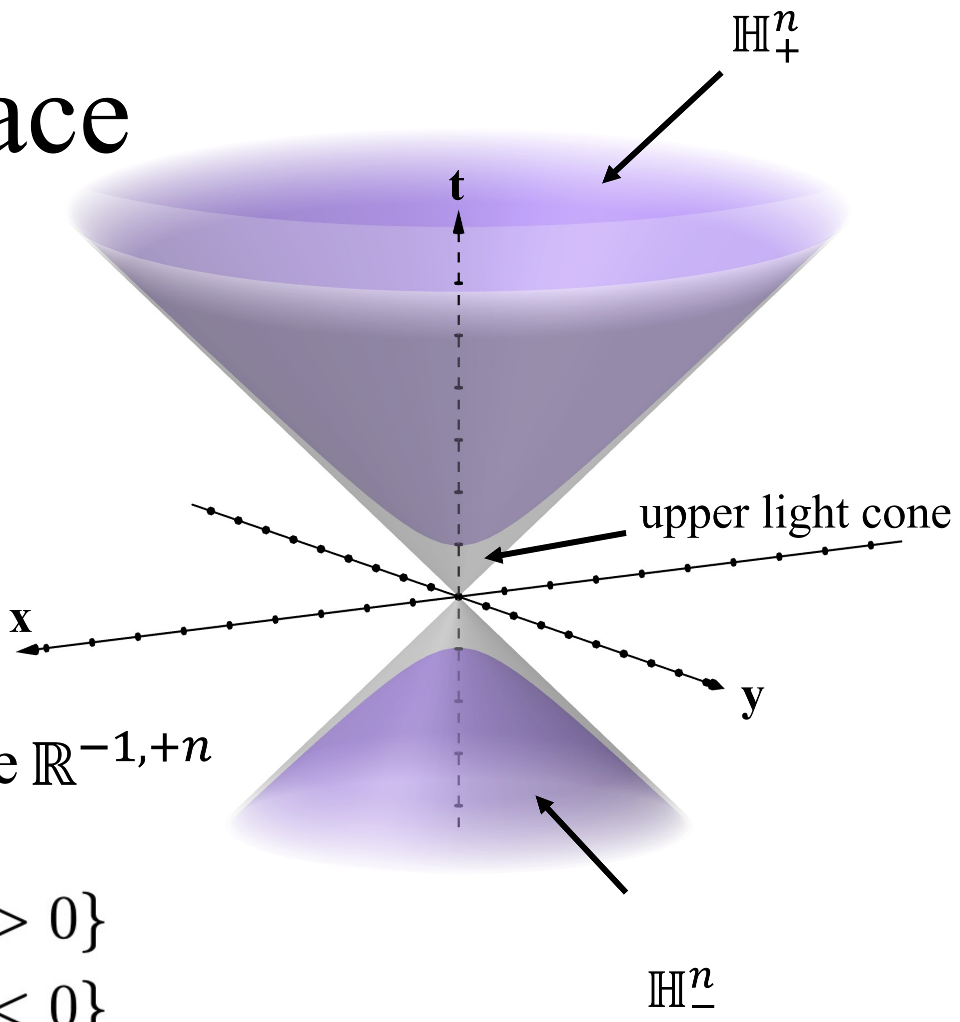
Angels and Devils M.C. Escher, 1960.

$\approx$



Order-4 Hexagonal Tiling.

Motivation • Hyperbolic Space



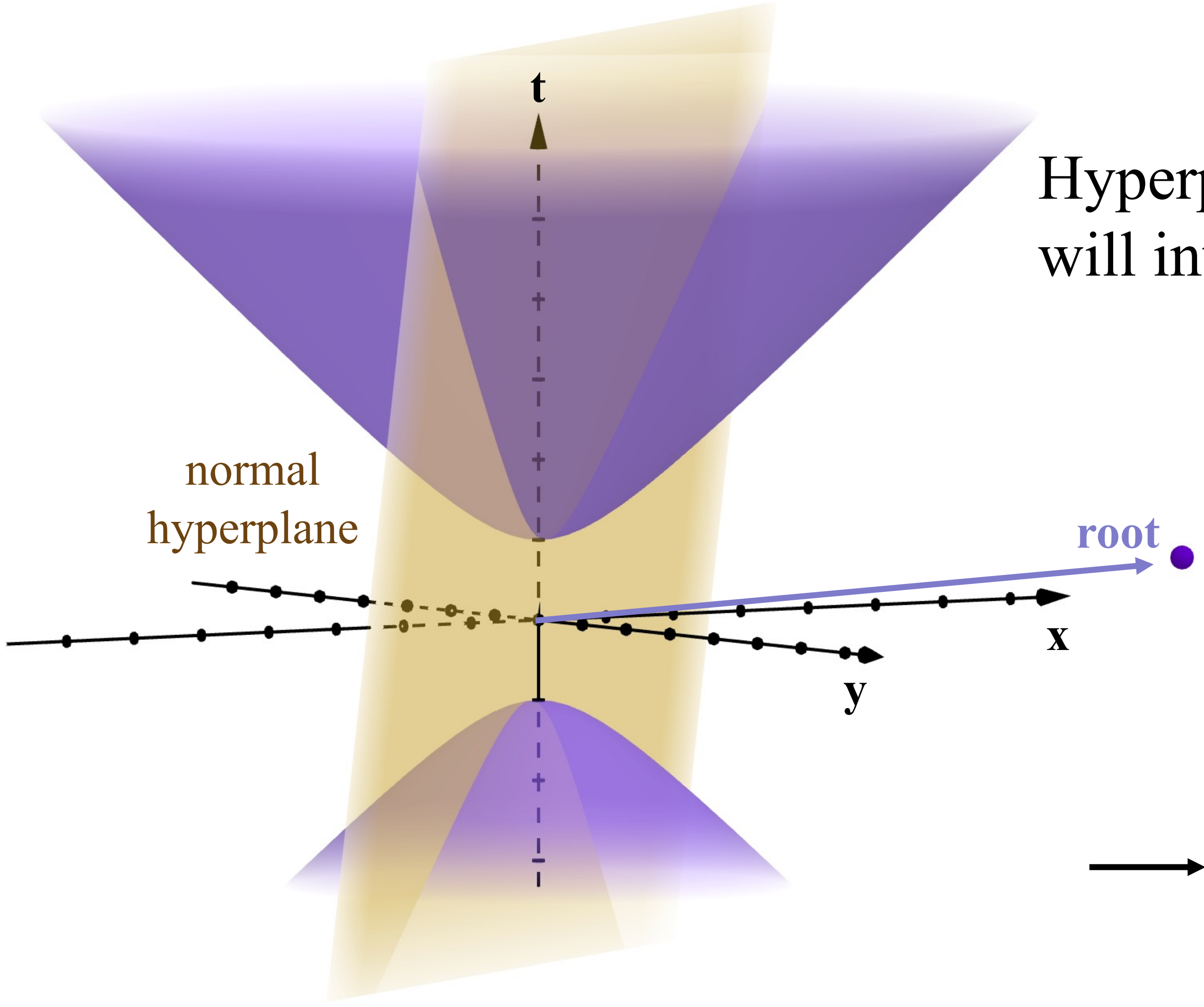
Why study Lorentzian reflection symmetries?
This has already been done! *(Partially)*

Can embed Hyperbolic space $\mathbb{H}^n$ in Minkowski space $\mathbb{R}^{-1,+n}$

i.e. upper sheet $\mathbb{H}^n_+ = \{\vec{v} \in \mathbb{R}^{-1,+n} \mid \langle \vec{v}, \vec{v} \rangle = -1, v^t > 0\}$
lower sheet $\mathbb{H}^n_- = \{\vec{v} \in \mathbb{R}^{-1,+n} \mid \langle \vec{v}, \vec{v} \rangle = -1, v^t < 0\}$

Choose upper sheet i.e. $\mathbb{H}^n = \mathbb{H}^n_+$

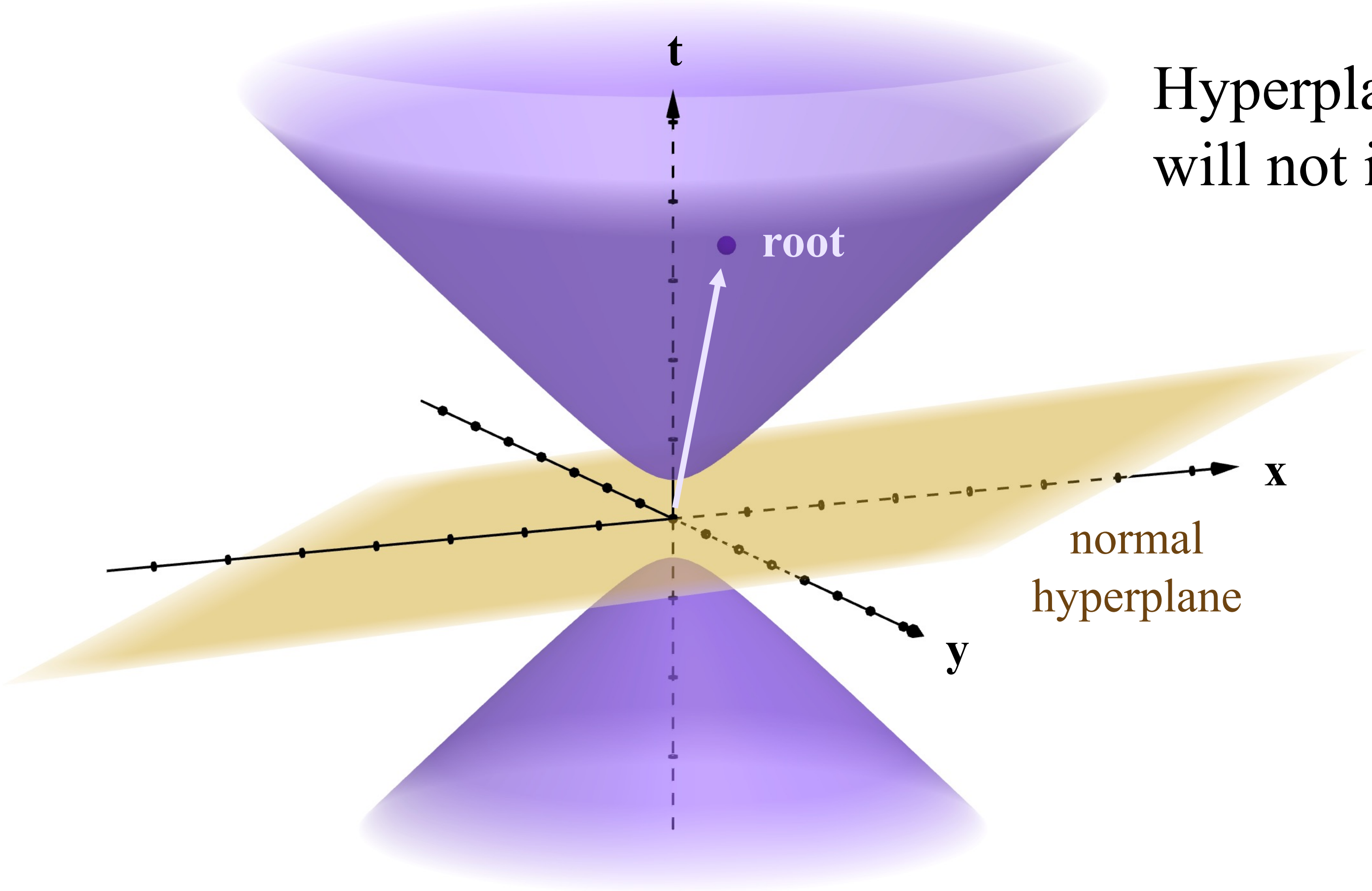
Motivation • Hyperbolic Space



Hyperplanes in $\mathbb{R}^{-1,+n}$ normal to **spacelike** roots will intersect $\mathbb{H}^n$

⟶ Reflections in $\mathbb{R}^{-1,+n}$ induce reflections in $\mathbb{H}^n$

Motivation • Hyperbolic Space

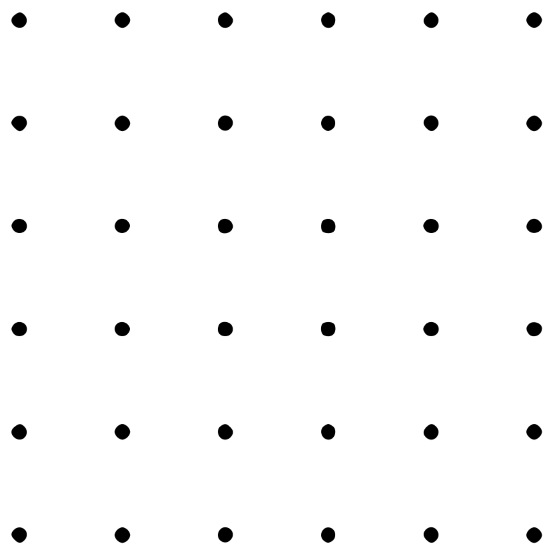


Hyperplanes in $\mathbb{R}^{-1,+n}$ normal to **timelike** roots will not intersect $\mathbb{H}^n$

Minkowski! • Examples

Let’s consider both spacelike **and timelike** reflections in $\mathbb{R}^{-1,+n}$

e.g. Integer Square Lattice in 1+1 Minkowksi



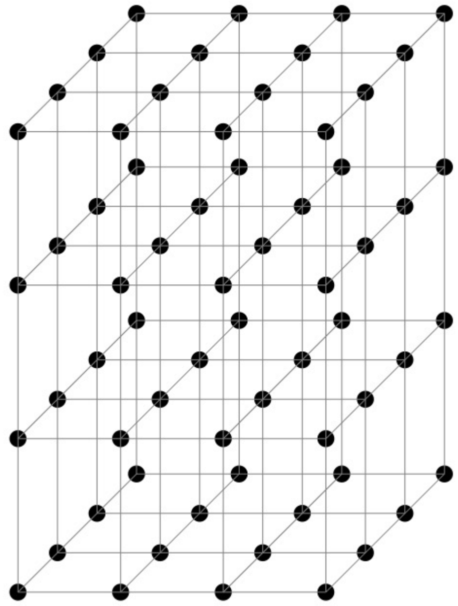
Only roots are t = 1, x = 0
t = 0, x = 1

Fundamental Domain is a 1x1 Square

Minkowski! • Examples

Let's consider both spacelike **and timelike** reflections in $\mathbb{R}^{-1,+n}$

e.g. Integer Cubic Lattice in 2+1 Minkowksi



Roots are integer points with norm +1, +2, –1, –2

⟶ Point group is infinite i.e. infinitely many mirrors through origin!

Preliminary Result • Too Many Mirrors!

Result For a collection of roots corresponding to mirrors through the origin in a Lorentzian space, a point group generated by reflections about these roots is infinite, unless either

(i) The induced metric in the space spanned by the roots is semi-definite.

(ii) The induced metric in the space spanned by the roots is indefinite, but for each pair of roots that span an indefinite space, such roots are orthogonal.

Basic Intuition = Hyperbolic "angles" are not bounded.

Kaleidoscopic Symmetry?

In indefinite spaces …

Crystallographic Symmetry $\not\leftrightarrow$ ***Kaleidoscopic Symmetry***

… and …

too many mirrors *(unless group is semidefinite or reducible)*

$\longrightarrow$ How can we have Kaleidoscopic Symmetry in indefinite spaces?

Future Work

What can be concluded from the above preliminary results?

Most likely …

Everything is "boring" = Kaleidoscopic reflection groups in Lorentzian space must be **reducible** to orthogonal definite **Euclidean** kaleidoscopic groups.

Thank You!

Big thanks to ***Latham*** for introducing me to these things and for the exciting math!

To ***Prof Henry Cohn*** for discussions.

To ***PSI Program*** and ***PSI friends*** for an unforgettable Master's program!

more

Crystallographic Symmetry!

Important concepts underlying above discussion.

A group Γ is ***crystallographic*** if it stabilizes some lattice Λ, i.e. $\gamma\Lambda = \Lambda$ for all $\gamma \in \Gamma$

Theorem Angles between reflecting hyperplanes in a crystallographic group can only be $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$.

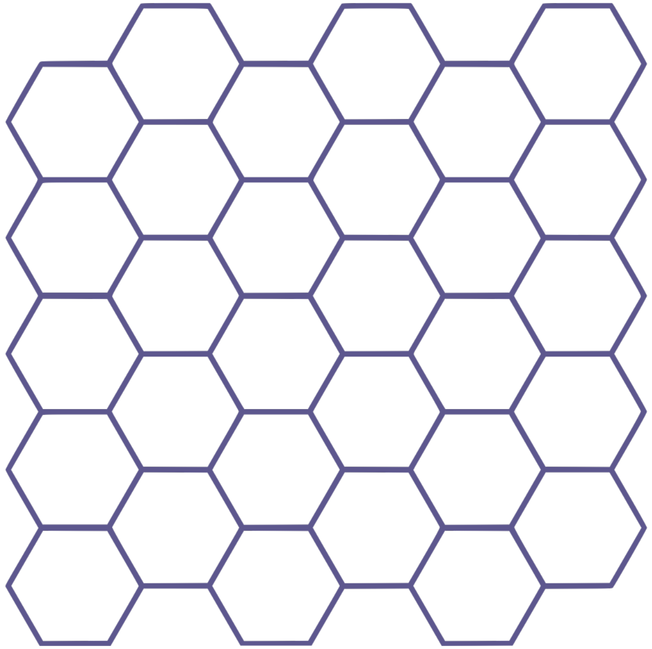
Fun Fact: *This is also the condition that guarantees elements in the Cartan Matrix for a finite semisimple Lie Group are integers!*

Euclidean • Preliminary

Question: What polygons tile the plane (2D Euclidean space)?

e.g. can a **hexagon** tile the plane?

→ ***yes!***

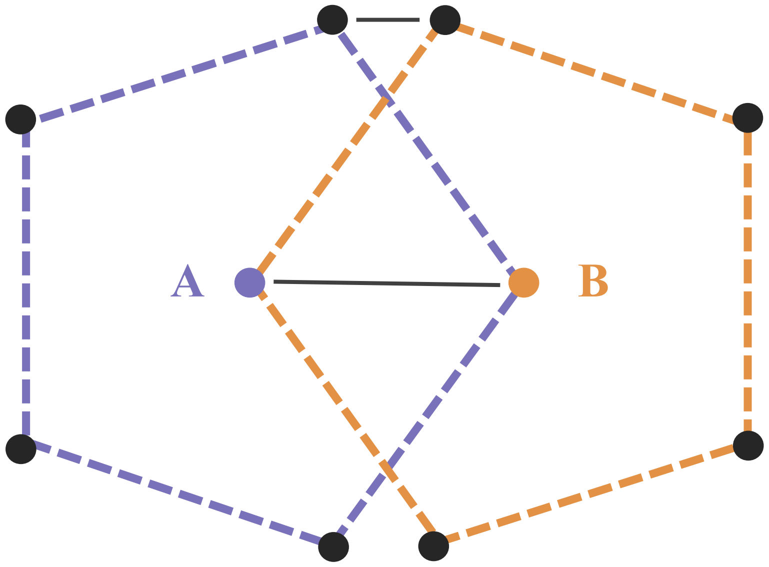


Euclidean • Preliminary

Question: What polygons tile the plane (2D Euclidean space)?

e.g. can a **pentagon** tile the plane?

→ ***no!***



Fundamental Domains

Understanding fundamental domains $\longleftrightarrow$ Classification of **irreducible finite Coxeter groups**

Fundamental Domain	Volume in space V bounded by mirror hyperplanes that is minimal (not further divided by operations in Γ)
Minimality of Fundamental Domain	Require $\langle \vec{r_i}, \vec{r_j} \rangle \leq 0$ for each distinct root of mirrors bounding fundamental domain
Irreducible	Mirrors defining the fundamental domain cannot be split into two disjoint sets that are mutually orthogonal.

Fundamental Domains

Understanding fundamental domains $\longleftrightarrow$ Classification of **irreducible finite Coxeter groups**

Minimality condition + Irreducibility Condition
$\longrightarrow$ no more than $n+1$ mirrors in n dimension Euclidean Space

Count number of mirrors:

Less than n mirrors	n mirrors	$n+1$ mirrors
Unbounded Fundamental Domain	***Spherical "Triangle"!***	***Euclidean "Triangle"!***

Preliminary Result • Fundamental Domains

Result For a fundamental domain in an indefinite space bounded by mirror hyperplanes to not be further subdivided upon reflection, need

(i) All spacelike and timelike roots of mirrors are orthogonal

(ii) For spacelike roots, $\langle \vec{r}_i, \vec{r}_j \rangle \leq 0$

(iii) For timelike roots, $\langle \vec{r}_i, \vec{r}_j \rangle \geq 0$

Ψ

Preliminary Result • Fundamental Domains

Recall *Minimality condition* $\langle \vec{r}_i, \vec{r}_j \rangle \leq 0$ *+ Irreducibility Condition*

$\longrightarrow$ *no more than* $n+1$ *mirrors in* n *dimension Euclidean Space*

This is no longer true in indefinite spaces!

Result Minimality condition $\langle \vec{r}_i, \vec{r}_j \rangle \leq 0$ for spacelike roots + Irreducibility Condition

$\longrightarrow$ arbitrarily many roots/mirror satisfy this if roots span indefinite subspace.

Result Minimality condition $\langle \vec{r}_i, \vec{r}_j \rangle \geq 0$ for timelike roots + Irreducibility Condition

$\longrightarrow$ arbitrarily many roots/mirror satisfy this if roots span indefinite subspace.

Future Work

What can be concluded from the above preliminary results?

Either …

Everything is “boring”	=	Kaleidoscopic reflection groups in Lorentzian space must be reducible to orthogonal definite Euclidean kaleidoscopic groups.

… or …

Things are weird	=	Things are weird when you allow roots to span indefinite spaces